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The asymmetric Graetz problem in channel flow

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NOMENCLATURE

A	coefficient in eigenfunction expansion of concentration
B	channel halfwidth [cm]
c_b	bulk concentration of species i [mol cm^{-3}]
c_i	concentration of species i [mol cm^{-3}]
c_o	surface concentration of species i [mol cm^{-3}]
D_i	diffusion coefficient of species i [cm^2 s^{-1}]
K	constant defined in equation (16)
Nu	Nusselt number, $\partial c_i/\partial y\|_{y=\pm B} 2B/(c_b-c_o)$
v_x	component of velocity in the axial direction [cm s^{-1}]
x	axial coordinate [cm]
y	normal coordinate, measured from center of the channel [cm].

Greek symbols

ζ	dimensionless axial coordinate defined in equation (4)
θ	dimensionless concentration defined in equation (2)
λ	eigenvalue in equation (11)
ξ	dimensionless normal coordinate defined in equation (3).

Subscripts

b	refers to bulk solution
i	refers to a particular species in solution
k	summation index in eigenfunction expansion (see equation (10))
o	refers to wall surface
-1	wall located at $\xi=-1$
1	wall located at $\xi=1$
0	refers to asymptotic solution of Sellars *et al.* [5].

Superscripts

*	dummy variable of integration.

INTRODUCTION

THE PROBLEM of mass transfer to fluids in laminar flow in ducts appears in many engineering applications. This problem has been solved for some special cases [1–11] but here we shall consider the case of a flat duct, or channel, where the surface-concentration boundary conditions are arbitrary and may differ on the two channel walls. This problem is of interest when the channel gap is thinner than the diffusion-boundary-layer thickness. In such cases, the fluxes at the two channel walls are not independent.

Since the detailed concentration profile within the flowing fluid is generally not needed, we shall emphasize the distribution of flux along the channel walls. Given this flux distribution, one can calculate the average concentration at the exit by performing an overall material balance.

To obtain the distribution of flux along the two channel walls, Duhamel's superposition principle [12, 13] may be used to treat the nonlinear concentration boundary conditions.

PROBLEM STATEMENT

For laminar flow in a channel, with negligible axial diffusion, the dimensionless convective diffusion equation is

$$(1-\xi^2)\frac{\partial \theta}{\partial \zeta}=\frac{\partial^2 \theta}{\partial \xi^2} \tag{1}$$

where

$$\theta=\frac{c_i-c_b}{c_o-c_b} \tag{2}$$

$$\xi=\frac{y}{B} \tag{3}$$

$$\zeta=x\frac{D_i}{\frac{3}{2}B^2\langle v_x\rangle}. \tag{4}$$

If the boundary conditions are arbitrary, Duhamel's superposition theorem may be used to write the flux in terms of the solution to the problem with a step-function concentration boundary condition on one wall. For example, if wall '−1' is located at $\xi=-1$, and wall '1' is located at $\xi=1$, then the flux of species i at wall '−1' is

$$N_{i,-1}(x)=-\frac{D_i}{B}\int_0^x \left.\frac{\mathrm{d}c_{i,-1}}{\mathrm{d}x}\right|_{x^*}\frac{\partial\theta}{\partial\xi}(\zeta-\zeta^*,\xi=-1)\,\mathrm{d}x^* + \frac{D_i}{B}\int_0^x \left.\frac{\mathrm{d}c_{i,1}}{\mathrm{d}x}\right|_{x^*}\frac{\partial\theta}{\partial\xi}(\zeta-\zeta^*,\xi=1)\,\mathrm{d}x^* \tag{5}$$

where $c_{i,-1}$ is the surface concentration of species i at $\xi=-1$ and $c_{i,1}$ is the concentration of species i at $\xi=1$. The flux (in the $+\xi$-direction) of species i at wall '1' is obtained by reversing the wall subscripts and the signs. In equation (5), $\theta(\zeta,\xi)$ is the solution to equation (1) with boundary conditions

$$\theta=1 \quad \text{at} \quad \xi=-1 \tag{6}$$

$$\theta=0 \quad \text{at} \quad \xi=1 \tag{7}$$

$$\theta=0 \quad \text{at} \quad \zeta=0. \tag{8}$$

THE LÉVÊQUE APPROACH

In general, equation (1) must be solved numerically. If, however, the diffusion boundary layers are thin, an analytic solution for the flux can be obtained by assuming that the velocity profile is linear throughout the boundary layer. This approximation, known as the Lévêque approximation [2, 3, 14, 15], is not valid throughout a thin-gap channel, but it is useful for treating the entrance region, where the diffusion boundary layers are thin. Norris and Streid have solved this Lévêque problem for channel flow [6].

One can extend the range of applicability of the Lévêque solution by writing a Lévêque series for the Nusselt number. This has been done for the Graetz problem in a tube [3, 15], and a similar procedure may be used for a channel to give

$$Nu=-2\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=-1}=1.35659745\zeta^{-1/3}-0.2-0.060733452\zeta^{1/3}. \tag{9}$$

THE GRAETZ APPROACH

To treat the downstream region, the Graetz approach (separation of variables) should be used. To calculate the dimensionless concentration

$$\theta = \frac{1}{2} - \frac{\xi}{2} + \sum_{k=1}^{\infty} A_k \, e^{-\lambda_k^2 \zeta} \, Y_k(\xi) \qquad (10)$$

the coefficients A_k, the eigenvalues λ_k, and the eigenfunctions Y_k are needed. The coefficients are obtained by using the orthogonality of the eigenfunctions with respect to the weight function $(1-\xi^2)$. The eigenvalues and eigenfunctions must be obtained numerically.

SOLUTION OF THE EIGENVALUE PROBLEM

One can rewrite the eigenvalue problem in a form convenient for numerical solution by realizing that the eigenvalues λ are constant. Thus

$$Y'' + \lambda^2(1-\xi^2)Y = 0 \qquad (11)$$

$$\frac{d\lambda^2}{d\xi} = 0. \qquad (12)$$

To solve equations (11) and (12), three boundary conditions are needed. The first two boundary conditions result from equations (6) and (7). The third boundary condition is a normalization condition

$$Y' = 1 \quad \text{at} \quad \xi = 1. \qquad (13)$$

The system of two ordinary differential equations with the three boundary conditions can be solved numerically using a finite-difference technique [15, 16].

COMBINING THE GRAETZ AND LÉVÊQUE SOLUTIONS

It should be noted that a truncated Graetz series (obtained from equation (10)) is accurate for large ζ, while the Lévêque series (equation (9)) is valid for small ζ. The value of ζ that divides the two regions is that value at which the ratio of the two asymptotic solutions is closest to unity. For example, if three terms are used in each series, the maximum error in the Nusselt number is 0.48% at $\zeta = 0.11$.

ASYMPTOTIC FORMS FOR LARGE EIGENVALUES

If greater accuracy than 0.48% is desired, then it is most efficient to add terms to the Graetz series. Therefore, it is useful to have simple asymptotic forms for calculating the higher eigenvalues and corresponding coefficients.

For the Graetz problem in a tube, Newman [15] extended the asymptotic forms of Sellars *et al.* [5] to achieve accuracy over a greater range of eigenvalues. Using a similar procedure, we devised an asymptotic form for the asymmetric Graetz problem (see Fig. 1)

$$\lambda = \lambda_0 + \frac{0.03254}{\lambda_0^{4/3}} - \frac{0.11}{\lambda_0^{14/3}} \qquad (14)$$

where

$$\lambda_0 = \frac{6k-1}{3} \quad \text{for} \quad k = 1, 2, \ldots \qquad (15)$$

is the asymptotic form obtained by modifying the method of Sellars *et al.* (see ref. [10]). The function in equation (14) may be used for $\lambda_6, \lambda_7, \ldots$, with a maximum error of $10^{-5}\%$.

The method of Sellars *et al.* predicts that the coefficients behave as $A_k = (-1)^{k+1}K/\lambda_k^{1/3}$ as λ_k becomes large, where

$$K = \frac{2^{4/3}\Gamma(2/3)}{3^{1/6}\Gamma(4/3)\pi} = 1.012787288. \qquad (16)$$

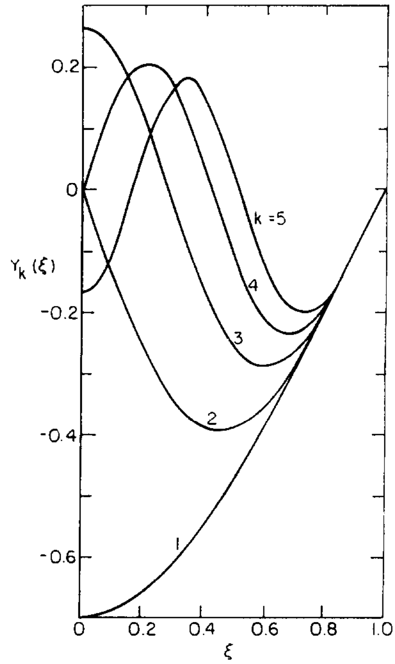

FIG. 1. Graetz functions for the asymmetric Graetz problem.

This asymptotic form was modified to

$$A_k = (-1)^{k+1}\left(\frac{K}{\lambda_k^{1/3}}\right)(1 + 0.03\lambda_k^{-4/3} - 0.03\lambda_k^{-8/3}). \qquad (17)$$

Equation (17) gives a maximum error of $4\times10^{-4}\%$ for $k \geq 6$.

Table 1 shows the comparison between the eigenvalues and coefficients as calculated by solving the eigenvalue problem and those calculated from the asymptotic form in equation (14). The accurate results of Brown [4] are also shown.

By using these asymptotic forms in a Graetz series with many terms, one can achieve a very high degree of accuracy in the Nusselt number.

EMPIRICAL APPROACH

Recall that the Lévêque solution is only applicable on the wall with the step change in concentration. To fit the region of small ζ on the opposite wall, it is more convenient to use an empirical function than it is to use a very large number of terms in the Graetz series.

The empirical function used here was derived by considering a simpler problem. If the fluid were in plug flow, rather than laminar flow, the mass-transfer problem would be analogous to the problem of transient heat conduction in a finite slab, where short time, t, is analogous to small ζ.

Based on the short-time solution for heat conduction in a finite slab, it is assumed that Nu is proportional to $e^{-b/\zeta}$. To match the behavior at low $1/\zeta$, a correction term of the form $c\,e^{-d/\zeta}$ can be added. The constants c and d were obtained by performing a least squares fit between ln Nu as calculated from a 100-term Graetz series and $\ln Nu = a - b/\zeta - c\,e^{-d/\zeta}$, for various specified values of d. The least squares fit was designed to weight the region of small ζ, where the empirical function is to be used. The resulting fitting function for small ζ is

$$Nu = -2\left.\frac{\partial\theta}{\partial\xi}\right|_{\xi=1} = \exp\left(0.9594 - 0.6069\frac{1}{\zeta} - 0.4512\,e^{-0.276/\zeta}\right). \qquad (18)$$

Table 1. Eigenvalues and coefficients for the asymmetric Graetz problem

	Eigenvalues, λ_k			Coefficients, A_k		
k	Brown [4]	Calculated	Asymptotic	Brown [4]	Calculated	Asymptotic
1	1.6815953222	1.6815953	1.6729924	0.8580866740	0.85808668	0.85807016
2		3.6722904	3.6721659		−0.65921993	−0.65931889
3	5.6698573459	5.6698573	5.6698540	0.5694628499	0.56946285	0.56949143
4		7.6688088	7.6688110		−0.51452221	−0.51453243
5	9.6682424625	9.6682424	9.6682441	0.4760654555	0.47606547	0.47606966
6		11.667894	11.667895		−0.44701873	−0.44702059
7	13.6676614426	13.667661	13.667662	0.4239737298	0.42397373	0.42397459
8		15.667496	15.667496		−0.40504973	−0.40505013
9	17.6673735653	17.667374	17.667374	0.3891087061	0.38910871	0.38910889
10		19.667279	19.667280		−0.37541429	−0.37541436

At $\zeta = 0.18$, the error between the 3-term Graetz series and the empirical function is 0.013%.

SUMMARY OF RESULTS

In summary, a convenient representation of the Nusselt number for the asymmetric Graetz problem has been obtained. For the wall with the step change in concentration, equation (9) is used for $\zeta < 0.11$ and

$$Nu = 1+2\sum_{k=1}^{3} |A_k|\, e^{-\lambda_k^2\zeta} \quad \text{for} \quad \zeta \geqslant 0.11 \tag{19}$$

where the eigenvalues and coefficients are listed in Table 1. For the wall without the step change, equation (18) is used for $\zeta < 0.18$ and

$$Nu = 1-2\sum_{k=1}^{3} A_k\, e^{-\lambda_k^2\zeta} \quad \text{for} \quad \zeta \geqslant 0.18. \tag{20}$$

It has been proposed that this solution may be used in a superposition integral to determine the wall flux in problems where the channel wall concentrations are arbitrary and may differ on the two walls.

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